Name ________________________ ID3 _ _ _

ECS 452: Digital Communication Systems **2017/2**

HW 3 — Due: Mar 9, 4 PM

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Instructions

(a) This assignment has 9 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Continue from Problem 2 of HW2. Consider a BSC whose crossover probability for each bit is $p = 0.35$. Suppose $P[X = 0] = 0.45$.

(a) Find the MAP detector and its error probability.

(b) Find the ML detector and its error probability.

Problem 2. Continue from Problem 3 of HW2. Consider a BAC whose $Q(1|0) = 0.35$ and $Q(0|1) = 0.55$. Suppose $P[X = 0] = 0.4$.

(a) Find the MAP detector and its error probability.

(b) Find the ML detector and its error probability.

Problem 3. Continue from Problem 4 of HW2. Consider a DMC whose $\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{1, 2, 3\}$, and $\mathbf{Q} = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$. Suppose the input probability vector is $\underline{\mathbf{p}} = [0.2, 0.4, 0.4]$.

(a) Find the MAP detector and its error probability.

(b) Find the ML detector and its error probability.

Problem 4. Consider a repetition code with a code rate of 1/5. Assume that the code is used with a BSC with a crossover probability $p = 0.4$. Find the ML detector and its error probability.

Derivation:

$$Q(\underline{y}|\underline{x}) = p^{d(\underline{x},\underline{y})}(1-p)^{n-d(\underline{x},\underline{y})} = (1-p)^n \left(\frac{p}{1-p}\right)^{d(\underline{x},\underline{y})}$$

$\frac{p}{1-p} < 0.5$ ⇒ use min dist. decoder

s	$\underline{x}$
0	00000
1	11111

majority voting

อธิบาย... MAP decoder

extended Q matrix

$\underline{x} \backslash \underline{y}$	...	11010	...	
00000	...	$(0.4^3)(0.6)^2$	...	× 0.8
11111	...	$(0.4)^2(0.6)^3$	...	× 0.2

extended P matrix

Problem 5. A channel encoder map blocks of two bits to five-bit (channel) codewords. The four possible codewords are 00000, 01000, 10001, and 11111. A codeword is transmitted over the BSC with crossover probability $p = 0.1$.

(a) What is the minimum (Hamming) distance d_{min} among the codewords?

(b) Suppose the codeword $\underline{\mathbf{x}} = 10001$ was transmitted. What is the probability that the receiver observes $\underline{\mathbf{y}} = 01001$ at the output of the BSC.

(c) Assume that all four codewords are equally likely to be transmitted. Suppose the receiver observes 01001 at the output of the BSC. What is the most likely codeword that was transmitted?

Problem 6. Consider random variables X and Y whose joint pmf is given by

$$p_{X,Y}(x,y) = \begin{cases} c\,(x+y), & x \in \{1,3\} \text{ and } y \in \{2,4\}, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the following quantities.

(a) c

(b) $H(X,Y)$

(c) $H(X)$

(d) $H(Y)$

(e) $H(X|Y)$

(f) $H(Y|X)$

(g) $I(X;Y)$

Problem 7. Consider a pair of random variables X and Y whose joint pmf is given by

$$p_{X,Y}(x,y) = \begin{cases} 1/15, & x=3, y=1, \\ 2/15, & x=4, y=1, \\ 4/15, & x=3, y=3, \\ \beta, & x=4, y=3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant β. $= \frac{8}{15}$

(b) Are X and Y independent? $\Rightarrow$ Yes

iff $p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$ — True for all possible x,y

$P =$

$x \backslash y$	1	3	$\Sigma \rightarrow$ $p_X(x) \equiv p(x)$
3	1/15	4/15	1/3
4	2/15	8/15	2/3
$\Sigma \downarrow$ $q(y) \equiv p_Y(y)$	1/5	4/5	

$\Rightarrow X \perp\!\!\!\perp Y$

(c) Evaluate the following quantities.

(i) $H(X) = -\frac{1}{3}\log_2\frac{1}{3} - \frac{2}{3}\log_2\frac{2}{3}$

(ii) $H(Y)$

(iii) $H(X,Y)$ = H(X) + H(Y) − ~~I(X;Y)~~ 0

(iv) $H(X|Y)$ = H(X) − ~~I(X;Y)~~ 0

(v) $H(Y|X)$ = H(Y) − ~~I(X;Y)~~ 0

(vi) $I(X;Y)$ = 0

Extra Questions

Here are some optional questions for those who want more practice.

Problem 8. Consider a repetition code with a code rate of 1/5. Assume that the code is used with a BSC with a crossover probability $p = 0.4$.

(a) Suppose the info-bit S is generated with

$$P[S = 0] = 1 - P[S = 1] = 0.4.$$

Find the MAP detector and its error probability.

(b) Assume the info-bit S is generated with

$$P[S = 0] = 1 - P[S = 1] = 0.45.$$

Suppose the receiver observes 01001.

(i) What is the probability that 0 was transmitted? (Do not forget that this is a conditional probability. The answer is not 0.45 because we have some extra information from the observed bits at the receiver.)

(ii) What is the probability that 1 was transmitted?

(iii) Given the observed 01001, which event is more likely, $S = 1$ was transmitted or $S = 0$ was transmitted? Does your answer agree with the majority voting rule for decoding?

(c) Assume that the source produces source bit S with

$$P[S = 0] = 1 - P[S = 1] = p_0.$$

Suppose the receiver observes 01001.

(i) What is the probability that 0 was transmitted?

(ii) What is the probability that 1 was transmitted?

(iii) Given the observed 01001, which event is more likely, $S = 1$ was transmitted or $S = 0$ was transmitted? Your answer may depend on the value of p_0. Does your answer agree with the majority voting rule for decoding?

Problem 9. A channel encoder map blocks of two bits to five-bit (channel) codewords. The four possible codewords are 00000, 01000, 10001, and 11111. Assume that the four codewords are not equally likely. Suppose 11111 is transmitted more frequently with probability 0.7. The other three codewords are transmitted with probability 0.1 each.

A codeword was transmitted over the BSC with crossover probability $p = 0.1$. Suppose the receiver observes 01001 at the output of the BSC. What is the most likely codeword that was transmitted?

$$\arg\max_{\underline{x}} P[\underline{X} = \underline{x} \mid \underline{Y} = 01001] = \frac{P[\underline{X} = \underline{x}, \underline{Y} = 01001]}{P[\underline{Y} = 01001]}$$

Extended Q matrix

$\underline{x} \backslash \underline{y}$	01001		Extended P matrix: 01001
00000	$(0.1)^2(0.9)^3$	×0.1 →	0.000729
01000	$(0.1)^1(0.9)^4$	×0.1 →	0.006561
10001	$(0.1)^2(0.9)^3$	×0.1 →	0.000729
11111	$(0.1)^3(0.9)^2$	×0.7 →	0.000567

Problem 10. Consider a DMC whose samples of input X and output Y are recorded as row vectors $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ in the file `HW_DMC_Channel_Data.mat`. Write MATLAB script which uses the recorded information to estimate the quantities below. Note that most of these can be solved by applying appropriate parts of the codes shown in class.

(a) The support $\mathcal{X}$ of X.

(b) The support $\mathcal{Y}$ of Y.

(c) The row vector $\underline{\mathbf{p}}$ which contains the pmf of X.

(d) The $\mathbf{Q}$ matrix.

(e) The row vector $\underline{\mathbf{q}}$ which contains the pmf of Y. Do this using two methods:

 (i) Count directly from the observed values of Y.

 (ii) Use the estimated values of $\underline{\mathbf{p}}$ and $\mathbf{Q}$.

(f) The error probability when the naive decoder is used. Do this using two methods:

 (i) Directly construct $\hat{\underline{\mathbf{x}}}$ from $\underline{\mathbf{y}}$. Then, compare $\hat{\underline{\mathbf{x}}}$ and $\underline{\mathbf{x}}$.

 (ii) Use the estimated values of $\underline{\mathbf{p}}$ and $\mathbf{Q}$.

(g) The error probability when the MAP decoder is used. Do this using two methods:

 (i) First find the MAP decoder table using the estimated values of $\underline{\mathbf{p}}$ and $\mathbf{Q}$. Then, construct $\hat{\underline{\mathbf{x}}}$ from $\underline{\mathbf{y}}$ according to the decoder table. Finally, compare $\hat{\underline{\mathbf{x}}}$ and $\underline{\mathbf{x}}$.

 (ii) Use the estimated values of $\underline{\mathbf{p}}$ and $\mathbf{Q}$ to directly calculate the error probability.

(h) The error probability when the ML decoder is used. Do this using two methods:

 (i) First find the ML decoder table using the estimated value of $\mathbf{Q}$. Then, construct $\hat{\underline{\mathbf{x}}}$ from $\underline{\mathbf{y}}$ according to the decoder table. Finally, compare $\hat{\underline{\mathbf{x}}}$ and $\underline{\mathbf{x}}$.

 (ii) Use the estimated values of $\underline{\mathbf{p}}$ and $\mathbf{Q}$ to directly calculate the error probability.